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Scalar-Tensor Theory with Torsion and Stellar Structure

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The modified Lane–Emden equation with an additional force, based on the scalar–tensor theory with torsion, is found. The influence of an additional intermediate-range force on stellar structure is investigated.

1. INTRODUCTION

Some time ago O'Hanlon (1972) suggested that the existence of an additional force is possible, namely, that the Newtonian gravitational potential is modified as

$$U(r) = -\frac{MG_\infty}{r}(1 + \mu e^{-\lambda r}) \tag{1}$$

where μ and λ^{-1} are the strength and the range of the additional force, respectively. Although the restriction on the additional force given by experiment laboratory is $|\mu| \leq 10^{-3} \sim 10^{-4}$ (Stubbs *et al.*, 1987), astrophysical and cosmological analysis shows the possibility of larger μ (Frieman *et al.*, 1991).

In our previous work (Xu *et al.*, 1991a,b), the additional force is explained as a manifestation of the torsion in the Riemann–Cartan spacetime U_4 with the aid of a scalar-tensor theory with torsion suggested by us. In this paper, we discuss the influence of the additional force on stellar structure based the scalar-tensor theory with torsion. In the next section, we briefly review the scalar-tensor model with torsion and the field equation.

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2. MODEL AND FIELD EQUATION

In the scalar-tensor model with torsion, the variational principle is (Xu *et al.*, 1991a,b)

$$\delta \int [\varphi R + kL + \varepsilon(\varphi - \varphi_0)^2]\sqrt{-g}\, d^4x = 0 \tag{2}$$

where k is a constant, ε is a coupling parameter, φ is the scalar function, φ_0 is the constant background value for the scalar-field φ, and L is the Lagrangian density, which clearly does not include φ, for matter. R is the curvature scalar in the Riemann–Cartan spacetime U_4 and can be written as follows (Xu, 1989):

$$R = R(\{\bullet\}) + g^{ij}T^l_{kj}K^k_{il} - \frac{4}{\sqrt{-g}}[\sqrt{-g}S^{ij}_j]_{,i} \tag{3}$$

In which $R(\{\bullet\})$ is the curvature scalar in the Riemann spacetime V_4, namely, the curvature scalar with respect to the Christoffel symbol. The comma used as an index indicates the usual derivative. Here

$$K^k_{ij} = -S^k_{ij} + S^k_{ij} + S^k_{ji} \tag{4}$$

is the contorsion tensor and

$$T^k_{ij} = S^k_{ij} + \delta^k_i S^l_{jl} - \delta^k_j S^l_{il} \tag{5}$$

is the modified torsion tensor. S^k_{ij} is the torsion tensor and is defined as

$$S^k_{ij} = \tfrac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) \tag{6}$$

where Γ^k_{ij} is the connection coefficient in U_4. Taking the torsion tensor as

$$S^k_{ij} = \frac{b}{2}\varphi^{-1}(\varphi_{,j}\delta^k_i - \varphi_{,i}\delta^k_j) \tag{7}$$

where b is a parameter which is independent of the spacetime point, we find that equation (3) becomes

$$R = R(\{\bullet\}) - \omega\varphi^{-2}\varphi^{,k}\varphi_{,k} + \frac{6b}{\sqrt{-g}}\varphi^{-1}(\sqrt{-g}\varphi^{,k})_{,k} \tag{8}$$

In which $\omega = 6b(b+1)$ is a new parameter. Substituting (8) into (2) and omitting the divergent term, we get

$$\delta \int [\varphi R(\{\bullet\}) - \omega\varphi^{-1}\varphi^{,k}\varphi_{,k} + \varepsilon(\varphi - \varphi_0)^2 + kL]\sqrt{-g}\, d^4x = 0 \tag{9}$$

By varying g_{ij} and φ in equation (9), respectively, we find the field equations

$$G_{ij}(\{\bullet\}) = R_{ij}(\{\bullet\}) - \tfrac{1}{2} g_{ij} R(\{\bullet\})$$
$$= \varphi^{-1}(\varphi_{,i|j} - g_{ij}\Box\varphi)$$
$$+ \omega\varphi^{-2}(\varphi_{,i}\varphi_{,j} - \tfrac{1}{2} g_{ij}\varphi^{,k}\varphi_{,k}) + \tfrac{1}{2}\varepsilon g_{ij}\varphi^{-1}(\varphi - \varphi_0)^2 + \tfrac{1}{2} k\varphi^{-1} T_{ij} \quad (10)$$

$$\Box\varphi + \frac{2\varepsilon\varphi_0}{2\omega + 3}(\varphi - \varphi_0) - \frac{k}{2(2\omega + 3)} T = 0 \quad (12)$$

where $R_{ij}(\{\bullet\})$ is the Ricci tensor with respect to the Christoffel symbol. $\Box\varphi = g^{ij}\varphi_{,i|j}$. The vertical bar denotes the covariant derivative using only the Christoffel symbol of the metric. According to the Bianchi identity, the Einstein tensor $G^{ij}(\{\bullet\})$ satisfies the identity

$$G^{ij}(\{\bullet\})^{j} = 0 \quad (12)$$

The energy-momentum tensor of matter T_{ij} is defined as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g}L)}{\partial g^{ij}} \quad (13)$$

$T = g^{ij}T_{ij}$. Using equations (10)–(12), we find that

$$T^{ij}_{\ |j} = 0 \quad (14)$$

3. THE WEAK-FIELD LINEAR APPROXIMATE SOLUTIONS

For a weak field, we write

$$g_{ij} = \eta_{ij} + h_{ij}, \qquad \varphi = \varphi_0 + \xi \quad (15)$$

where η_{ij} is the Minkowskian metric tensor. h_{ij} and ξ are small perturbations and they are computed to the linear first approximation only. Therefore, substituting (15) into (11), we get

$$-\nabla^2\xi + \frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} + \lambda^2\xi = \tfrac{1}{2} k\mu T \quad (16)$$

in which

$$\lambda^2 = \frac{2\varepsilon\varphi_0}{2\omega + 3} \qquad \text{and} \qquad \mu = \frac{1}{2\omega + 3}$$

The retarded-time solution of equation (10) is

$$\xi = \frac{k\mu}{8\pi} \int \frac{T}{r} e^{-\lambda r} \, d^3x \quad (17)$$

where T is to be evaluated at retarded time. Substituting (15) into (10) and introducing the coordinate condition

$$(h_{ij} - \tfrac{1}{2}\eta_{ij}h)_{,k}\eta^{jk} = \varphi_0^{-1}\xi_{,i} \tag{18}$$

we find that equation (10) becomes

$$-\nabla^2\alpha_{ij} + \frac{1}{c^2}\frac{\partial^2\alpha_{ij}}{\partial t^2} = -k\varphi_0^{-1}T_{ij} \tag{19}$$

where

$$\alpha_{ij} = h_{ij} - \tfrac{1}{2}\eta_{ij}h - \eta_{ij}\varphi_0^{-1}\eta \tag{20}$$

The retarded-time solution of equation (19) is

$$\alpha_{ij} = -\frac{k\varphi_0^{-1}}{4\pi}\int\frac{T_{ij}}{r}\,d^3x \tag{21}$$

From equations (17), (20), and (21), we get

$$\begin{aligned} h_{ij} &= \alpha_{ij} - \frac{1}{2}\eta_{ij}\alpha - \eta_{ij}\varphi_0^{-1}\xi \\ &= \frac{\alpha\varphi_0^{-1}}{4\pi}\left[-\int\frac{T_{ij}}{r}\,d^3x + \frac{1}{2}\eta_{ij}\int\frac{T}{r}(1 - \mu e^{-\lambda r})\,d^3x\right] \end{aligned} \tag{22}$$

For a stationary mass point of mass M, from equations (15) and (22), we obtain the weak-field approximate solutions

$$g_{44} = 1 + \frac{2U(r)}{c^2} \tag{23}$$

$$g_{\alpha\alpha} = -1 - \frac{kMc^2\varphi_0^{-1}}{8\pi r}(1 - \mu e^{-\lambda r}), \qquad \alpha = 1, 2, 3 \tag{24}$$

where

$$U(r) = -\frac{kMc^4\varphi_0^{-1}}{16\pi r}(1 + \mu e^{-\lambda r}) \tag{25}$$

Putting $k = 16\pi/c^4$ and with $\varphi_0^{-1} = G_\infty$, the Newtonian constant of gravitation for $r \to \infty$, we find that equation (25) becomes equation (1).

4. MODIFIED LANE–EMDEN EQUATION AND STELLAR STRUCTURE

For a static, spherically symmetrical perfect fluid with density $\rho(r)$, low pressure $p(r)$, and radius R, the nonzero components of the energy-momentum tensor are

$$T^{\alpha}_{\beta} = -p(r)\delta^{\alpha}_{\beta}, \qquad T^4_4 = \rho(r)c^2 \qquad (\alpha, \beta = 1, 2, 3) \tag{26}$$

Substituting (26) into (14), we obtain the equilibrium equation

$$p_{,\alpha} + \tfrac{1}{2}(p + \rho c^2)h_{44,\alpha} = 0 \tag{27}$$

Substituting (20) into (27), we get

$$\frac{1}{p + \rho c^2}\nabla p = -\tfrac{1}{2}(\nabla\alpha_{44} - \tfrac{1}{2}\eta_{44}\nabla\alpha - \eta_{44}\varphi_0^{-1}\nabla\xi) \tag{28}$$

Here ∇ is the three-dimensional Laplacian operator. Taking the divergence for the above equation, we find

$$\nabla \bullet \left(\frac{1}{p + \rho c^2}\nabla p\right) = -\frac{1}{2}\nabla^2\alpha_{44} + \frac{1}{4}\eta_{44}\nabla^2\alpha + \frac{1}{2}\eta_{44}\varphi_0^{-1}\nabla^2\xi \tag{29}$$

Substituting the static equations corresponding to (16) and (19) into (29), taking account of the low-pressure approximation $p << \rho c^2$, and putting $k = 16\pi/c^4$ and $\varphi_0^{-1} = G_\infty$, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{1}{\rho}\frac{d}{dr}p\right) = -4\pi G_\infty\rho(1 + \mu) + \frac{1}{2}G_\infty c^2\lambda^2\xi \tag{30}$$

We assume that the relationship between the pressure p and the density ρ is described by a polytropic equation

$$p = K\rho^{1+1/N} \tag{31}$$

where K is a constant, and N is the polytropic index. Substituting (31) into (30) and introducing new variables

$$\theta = \left(\frac{\rho}{\rho_0}\right)^{1/N} \tag{32}$$

$$x = \left[\frac{4\pi G_\infty}{K(N+1)}\rho_0^{1-1/N}\right]^{1/2} r = \frac{r}{\beta} \tag{33}$$

where

$$\beta = \left[\frac{K(N+1)}{4\pi G_\infty}\rho_0^{(1/N)-1}\right]^{1/2} \tag{34}$$

we obtain the modified Lane–Emden equation

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) = -(1+\mu)\theta^N + \frac{c^2\lambda^2}{8\pi\rho_0}\xi \tag{35}$$

where ρ_0 is the density at the center. The boundary conditions of equation (35) at the center are

$$\theta(0) = 1, \qquad \frac{d\theta}{dx}(0) = 0 \tag{36}$$

In the absence of the additional force, then $\mu = 0$ and $\xi = 0$, and equation (35) becomes the Lane–Emden equation in the Newtonian theory.

From equation (32), the static field equation corresponding to (16) may be rewritten as

$$-\frac{1}{\beta^2x^2}\frac{d}{dx}\left(x^2\frac{d\xi}{dx}\right) + \lambda^2\xi = \frac{8\pi}{c^2}\mu\rho_0\theta^N \tag{37}$$

Substituting (35) into (37), we get

$$\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right) - \beta^2\lambda^2\right]\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) + (1+\mu)\theta^N\right] = -\mu\lambda^2\beta^2\theta^N \tag{38}$$

we discuss two cases as follows:

Case 1. $N = 0$: We discuss a static uniform star with density ρ_0 and radius R. In this case, equation (37) has the exterior solution satisfying the continuity condition at the stellar surface

$$\xi(x) = \frac{8\pi\mu\rho_0}{\lambda^3c^2\beta x}[\lambda\beta x_0 \cosh(\lambda\beta x_0) - \sinh(\lambda\beta x_0)]e^{-\lambda\beta x} \tag{39}$$

and the interior solution

$$\xi(x) = \frac{8\pi\mu\rho_0}{\lambda^2c^2}\left[1 - \frac{1}{\lambda\beta x}e^{-\lambda\beta x_0}(1+\lambda\beta x_0)\sinh(\lambda\beta x)\right] \tag{40}$$

in which $x_0 = \beta^{-1}R$. From equations (35) and (40), the Lane–Emden equation for $N = 0$ is written as

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) + 1 = -\frac{\mu}{\lambda\beta x}(1+\lambda\beta x_0)e^{-\lambda\beta x_0}\sinh(\lambda\beta x) \tag{41}$$

This equation has the solution satisfying the conditions (36)

$$\theta(x) = 1 - \frac{1}{6}x_2 + \frac{\mu}{\lambda^2\beta^2}(1 + \lambda\beta x_0)e^{-\lambda\beta x_0}\left[1 - \frac{1}{\lambda\beta x}\sinh(\lambda\beta x)\right] \quad (42)$$

From the boundary condition at the stellar surface $\theta(x_0) = 0$, we get that

$$1 - \frac{1}{6}x_0^2 + \frac{\mu}{\lambda^2\beta^2}(1 + \lambda\beta x_0)e^{-\lambda\beta x_0}\left[1 - \frac{1}{\lambda\beta x_0}\sinh(\lambda\beta x_0)\right] = 0 \quad (43)$$

For the intermediate-range additional force, we may take the approximation $\lambda R >> 1$ and obtain the expression of the stellar radius

$$R = R_N\left(1 - \frac{\mu}{2\lambda^2\beta^2}\right)^{1/2} \approx R_N\left(1 - \frac{\mu}{4\lambda^2\beta^2}\right) \quad (44)$$

in which $R_N = \sqrt{6}\beta$ is the stellar radius in the Newtonian theory. The fractional change in radius, from equation (44), is

$$\frac{\delta R}{R_N} = \frac{R - R_N}{R_N} = -\frac{\mu}{4\lambda^2\beta^2} \quad (45)$$

Case 2. $N = 1$: Equation (38) is written as

$$\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right) - \lambda^2\beta^2\right]\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\theta}{dx}\right) + (1 + \mu)\theta\right] = -\mu\lambda^2\beta^2\theta \quad (46)$$

Equation (46) has the solution satisfying the boundary conditions (36)

$$\theta(x) = \frac{\sin(\omega x)}{\omega x} + C\Omega\left(\frac{\sinh(\Omega x)}{\Omega x} - \frac{\sin(\omega x)}{\Omega x}\right) \quad (47)$$

where

$$\begin{aligned}\omega^2 &= \tfrac{1}{2}\{1 + \mu - \lambda^2\beta^2 + [(1 + \mu - \lambda^2\beta^2)^2 + 4\lambda^2\beta^2]^{1/2}\} \\ \Omega^2 &= -\tfrac{1}{2}\{1 + \mu - \lambda^2\beta^2)^2 - [(1 + \mu - \lambda^2\beta^2)^2 + 4\lambda^2\beta^2]^{1/2}\}\end{aligned} \quad (48)$$

The constant C is determined by the zero-pressure boundary condition $\theta(x_0) = 0$ at the stellar surface

$$C = \frac{\sin(\omega x_0)}{\Omega\sin(\omega x_0) - \omega\sinh(\Omega x_0)} \quad (49)$$

Substituting (47) into (35), we obtain the interior solution of equation (37)

$$\xi = \frac{8\pi\rho_0}{\lambda^2 c^2 x}\left[(C\Omega - 1)\omega_\mu\frac{\sin(\omega x)}{\omega} + C\Omega_\mu\sinh(\Omega x)\right] \quad (50)$$

where

$$\omega_\mu = \omega^2 - 1 - \mu \qquad \Omega_\mu = \Omega^2 + 1 + \mu \tag{51}$$

The exterior solution of equation (37) may be written as

$$\xi = \frac{8\pi\rho_0 B}{\lambda^2 c^2 x} e^{-\lambda\beta x} \tag{52}$$

where B is an integral constant. Using the continuity of ξ and $d\xi/dx$ at x_0, we obtain the equation determining x_0 as follows:

$$\lambda\beta(\omega^2 + \Omega^2) + \omega_\mu\omega \cot(\omega x_0) + \Omega_\mu\Omega \coth(\Omega x_0) = 0 \tag{53}$$

For the intermediate-range additional force, taking the approximation $\lambda R >> 1$, the expression of the stellar radius is written as

$$R \approx \beta\pi \left(1 - \frac{\mu}{2\lambda^2\beta^2}\right) \tag{54}$$

Thus, the fractional change in radius is

$$\frac{\delta R}{R_N} = -\frac{\mu}{2\lambda^2\beta^2} \tag{55}$$

This result is the same as found by Glass *et al.* (1989) in another way.

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